Computational tools for hypergeometric L-functions

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These slides can be downloaded from https://kskedlaya.org/slides/.

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I acknowledge that my workplace occupies unceded ancestral land of the Kumeyaay Nation.

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Hypergeometric data

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$$P(\alpha;\beta)(z\frac{d}{dz})(y) = 0, \quad P(\alpha;\beta)(D) := z\prod_{i=1}^{n}(D+\alpha_i) - \prod_{j=1}^{n}(D+\beta_j-1).$$

The Hodge vector/motivic weight can be read from the **zigzag function**

$$Z_{\alpha,\beta}(x) := \#\{j : \alpha_j \leq x\} - \#\{j : \beta_j \leq x\}.$$

Hereafter we assume that α, β are **Galois-stable**,[1] meaning that the multiplicity of any $\frac{r}{s} \in \mathbb{Q}$ (in lowest terms) depends only on s.

[1] Otherwise we get motives defined only over some abelian extension of $\mathbb{Q}$.

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L-functions

For α, β Galois-stable, this variation of Hodge structures arises from a family of Chow motives $M^{\alpha,\beta}$ over $\mathbb{Q}$.

For any given $z \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$, the motive $M_z^{\alpha,\beta}$ has bad reduction[2] at these primes:

- **wild** primes p, at which α or β is not in $\mathbb{Z}_{(p)}^n$;
- **tame** primes p, which are not wild but either z or $z-1$ is not a p-adic unit.

For such z, we obtain an associated L-function; as usual this manifests as an Euler product

$$\prod_p L_{\alpha,\beta,z,p}(p^{-s})^{-1}, \qquad L_*(T) \in 1 + T\mathbb{Z}[T].$$

In general, we expect this to coincide with the L-function of some **automorphic form** for a certain reductive group defined in terms of the **monodromy group** of the Picard–Fuchs equation; this specializes to **modularity** when the group is GL_2.

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Computational tools

The goal of this talk is to survey the computational tools available for computing coefficients of a hypergeometric L-function (PARI/GP, Magma, SageMath, LMFDB, etc.) Which tool is best depends somewhat on the use case involved; here are a few examples.

- Computing **good Euler factors** up to some cutoff. This gives information about the **Sato–Tate distribution**.
- Computing **all Dirichlet coefficients** up to some cutoff. This gives information about **special values** or **analytic properties** (e.g., the analogue of the Riemann hypothesis).
- Computing **good Frobenius traces** up to some cutoff. This can be done at greater scale thanks to my work with Costa and Roe.
- The reverse problem: given an L-function, look for a match among hypergeometrics.

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A peek under the hood

All of these systems implement a formula for good/tame hypergeometric Frobenius traces by Beukers–Cohen–Mellit in terms of Gauss sums, as reinterpreted using the Gross–Koblitz formula by Cohen–Rodriguez Villegas-Watkins.

- Magma: Mark Watkins
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One complication is that **wild** Euler factors are not known in all cases (see work of Roberts–Rodriguez Villegas for partial results). However, given a guess for all bad Euler factors (and conductors), one can numerically verify using the approximate functional equation.

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Introduction

Let vecα, vecβ∈(C)^n be n-tuples (or multisets) of complex numbers. For arithmetic applications we will eventually take them to be rationals, and for purposes of monodromy will largely need only to consider them modulo 1.

Consider the generalised hypergeometric differential equation z(θ + α_1) ... (θ + α_n)F(z) =(θ + β_1 - 1) ... (θ + β_n - 1)F(z), qquadθ=z(d/dz), whose only singularities are regular at 0, 1, and ∞. For simplicity of exposition, we assume that the β's are distinct modulo 1, when a basis of solutions around z=0 is given by $z^{1 - \beta_i}$()$_n$F$_{n - 1}$ biggl((α_1 - β_i + 1, ..., α_n - β_i + 1atop β_1 - β_i + 1, ..v.., β_n - β_i + 1)biggm|z biggr) for i=1 ... n, and the ith term β_i - β_i + 1 is suppressed. The generalised hypergeometric function ()$_n$F$_{n - 1}$ is given by ()$_n$F$_{n - 1}$ biggl((a_1, ..., $a_n$atop b_1, ..., $b_{n - 1}$)biggm|z biggr)= $\sum_{k=0}$^∞((a_1)$_k$ ... (a_n)$_k$/(b_1)$_k$ ... ($b_{n - 1}$)$_k$) (z^k/k!), where the Pochhammer symbol is given by $(x)_k$=(x)(x + 1) ... (x + k - 1); the k! in the denominator of the above display can thus be thought of as $(1)_k$, which was the suppressed term. Note that shifting all the α and β by some fixed amount keeps the ()$_n$F$_{n - 1}$ expression the same, while only modifying the $z^{1 - \beta_i}$ term. Also, switching α and β can be envisaged in terms of the map z -> 1/z that swaps 0 and ∞.

A theorem of Pochhammer says that the above differential equation has (n - 1) independent *holomorphic* solutions around z=1. Let G denote the fundamental group of the thrice punctured Riemann sphere, and $V_{\text{vec}\alpha, \text{vec}\beta}$ the solution space around a base point. We have a monodromy representation M:G -> GL_n($V_{\text{vec}\alpha, \text{vec}\beta}$). Writing g_0, g_1, g_∞∈G for loops about 0,1,∞, we find that M(g_0) has eigenvalues $e^{- 2\pi i\beta_j}$ and M(g_∞) has eigenvalues $e^{2\pi i\alpha_j}$, implying that we are mainly concerned with vecα and vecβ only modulo 1. Indeed, one can note that if we take F=()$_n$F$_{n - 1}$(vec a, vec b|z) and vec x∈(Z)^n, vec y∈(Z)$^{n - 1}$, then generically ()$_n$F$_{n - 1}$(vec a + vec x, vec b + vec y|z) is a linear combination of rational functions times derivatives of F (this is a contiguity relation). Meanwhile, the above fact from Pochhammer implies that M(g_1) must have (n - 1) eigenvalues equal to 1 (all with independent eigenvectors), and so this element is a pseudo-reflection.

It turns out that if H⊆GL_n((C)) is generated by A and B with $AB^{- 1}$ a pseudo-reflection, the H-action on (C)^n is irreducible if and only if A and B have disjoint sets of eigenvalues. This is equivalent to all the α_i - β_j being nonintegral. Moreover, in his 1961 Amsterdam thesis, Levelt showed that, given any eigenvalues, there are (up to conjugacy) unique A and B realising these eigenvalues with $AB^{- 1}$ a pseudo-reflection. (Much of the above comes from notes of Beukers.)

For arithmetic purposes, one usually also desires that the eigenvalues be roots of unity and the sets of them be Galois-invariant. Thus we can specify hypergeometric data H by (say) two products of cyclotomic polynomials, these products being coprime and of equal degree. Given such an H, Rodriguez-Villegas conjectures the existence of a family of pure motives (defined over **Q**), for which the trace of Frobenius at good primes is given by a hypergeometric sum defined by Katz [Kat90] (see also [Kat96]). For each rational t != 0, 1, there should be a motive H_t whose L-function satisfies a functional equation of a prescribed type, with the Euler factors at good primes given in terms of Gauss sums (the bad Euler factors are less understood, and depend on deformation theory).

One can also relate such motives to more traditional objects in many cases. For instance, there is one hypergeometric datum in degree 1, which can be specified by α=[(1/2)] and β=[0], these being rationals corresponding to the second and first cyclotomic polynomials respectively. The L-function here corresponds to the quadratic field (Q)(Sqrt(t(t - 1))). In degree 2 there are 13 such data, of which 3 are of weight 0 (see below) and give Artin representations of number fields, while the other 10 are of weight 1, and yield elliptic curves (explicitly calculated by Cohen). An example in higher degree is α=[(1/5), (2/5), (3/5), (4/5)] and β=[0, 0, 0, 0], corresponding respectively to the 5th cyclotomic polynomial and the 4th power of the first cyclotomic polynomial, and this is associated to the Calabi-Yau quintic 3-fold given by x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5=5t$x_1x_2x_3x_4x_5$.

The weight w of a hypergeometric motive can be defined in terms of how much the α and β interlace (considered as roots of unity). In particular, if they are completely interlacing, then the weight is 0, and the resulting motive corresponds to an Artin representation. Write D(x)=#{α : α≤x} - #{β : β≤x}. Then w + 1=$\max_x$ D(x) - $\min_x$ D(x), so that the above 3-fold has weight 3 (from the four β's at 0). This weight controls how large the coefficients of the Euler factors will be.

The trace at q of a hypergeometric motive (for the parameter t) is given in terms of Gauss sums g_q over $(F)_q$. Associated to hypergeometric data is a `GammaArray`, and one defines G_q(r)=$\prod_v$ g_q(- rv)$^{\gamma_v}$, and also the `MValue` by M=$\prod_v$ $v^{v\gamma_v}$. For primes p with v_p(Mt)=0 the hypergeometric trace is then given by U_q(t)=(1/1 - q)biggl($\sum_{r=0}^{q - 2}$$\omega_p$(Mt)^r$Q_q$(r)biggr), where ω_p is the Teichmüller character and Q_q(r)=(- 1)^{m_0}$q^{D + m_0 - m_r}$$G_q$(r) where m_r is the multiplicity of (r/q - 1) in the β and D is a scaling parameter that involves the Hodge structure (one expression is m_0=w + 1 - 2D). One uses p-adic Γ-functions to expedite the computation of the above Gauss sums (indeed, the above gives the hypergeometric trace as a p-adic number, which one recognises as an integer via sufficiently high precision). The Euler factor is given by the standard recipe E_p(T)=exp(- $\sum_n$ U_{p^n}(t)T^n/n), and this is a polynomial that satisfies a local functional equation.

Notes

The parameter z is interchanged with $1/z$ relative to the standard normalization. This is also true in PARI/GP.

Some cacheing of p-adic Gamma functions is used to speed up the computation, especially when many computations are done in a single session.

There is a fair bit of extra functionality beyond simply computing L-functions. For instance, some matching with other Hasse-Weil L-functions (such as for elliptic curves) is implemented.

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Contents

Hypergeometric L-functions via PARI/GP: a snapshot

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Hypergeometric Motives

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Templates

A *hypergeometric template* is a pair of multisets (i.e., sets with possibly repeated elements) of rational numbers $(\alpha_1,\dots,\alpha_d)$ and $(\beta_1,\dots,\beta_d)$ having the same number of elements, and we set $A(x) = \prod_{1 \le j \le d}(x-e^{2\pi i\alpha_j})$, $B(x) = \prod_{1 \le k \le d}(x-e^{2\pi i\beta_k})$. We make the following assumptions:

* $\alpha_j-\beta_k \notin \mathbb{Z}$ for all j and k, or equivalently gcd(A,B) = 1.

* $\alpha_j \notin \mathbb{Z}$ for all j, or equivalently A(1) != 0.

* our template is *defined over* $\mathbb{Q}$, in other words $A,B \in \mathbb{Z}[x]$, or equivalently if some a/D with gcd(a,D) = 1 occurs in the α_j or β_k, then all the b/D modulo 1 with gcd(b,D) = 1 also occur.

The last assumption allows to abbreviate $[a_1/D,\dots,a_{\varphi(D)}/D]$ (where the a_i range in $(\mathbb{Z}/D\mathbb{Z})^*$) to [D]. We thus have two possible ways of giving a hypergeometric template: either by the two vectors $[\alpha_1,\dots,\alpha_d]$ and $[\beta_1,\dots,\beta_d]$, or by their denominators $[D_1,\dots,D_m]$ and $[E_1,\dots,E_n]$, which are called the *cyclotomic parameters*; note that $\sum_j\varphi(D_j) = \sum_k\varphi(E_k) = d$. A third way is to give the *gamma vector* (γ_n) defined by $A(X)/B(X) = \prod_n(X^n-1)^{\gamma_n}$, which satisfies $\sum_n n\gamma_n = 0$. To any such data we associate a hypergeometric template using the function `hgminit`; then the α_j and β_k are obtained using `hgmalpha`, cyclotomic parameters using `hgmcyclo` and the gamma vectors using `hgmgamma`.

To such a hypergeometric template is associated a number of additional parameters, for which we do not give the definition but refer to the survey *Hypergeometric Motives* by Roberts and Villegas, https://arxiv.org/abs/2109.00027: the degree d, the *weight* w, a *Hodge polynomial* P, a *Tate twist* T, and a normalizing M-factor $M = \prod_n n^{n\gamma_n}$. The `hgmparams` function returns [d,w,[P,T],M] . Example with cyclotomic parameters [5],[1,1,1,1]:

```
? H = hgminit([5]); \\ [1,1,1,1] can be omitted
? hgmparams(H)
%2 = [4, 3, [x^3+x^2+x+1,0], 3125]
? hgmalpha(H)
%3 = [[1/5, 2/5, 3/5, 4/5], [0, 0, 0, 0]]
? hgmcyclo(H)
%4 = [Vecsmall([5]), Vecsmall([1, 1, 1, 1])]
? hgmgamma(H)
%5 = Vecsmall([-5,0,0,0,1]) \\ A/B = (x^5-1) / (x-1)^5
```

Motives

A *hypergeometric motive* (HGM for short) is a pair (H,t), where H is a hypergeometric template and $t \in \mathbb{Q}^*$. To such a motive and a finite field $\mathbb{F}_q$ one can associate via an explicit but complicated formula an *integer* N(H,t; q), see Beukers, Cohen and Mellit, *Finite hypergeometric functions* Pure and Applied Math Quarterly 11 (2015), pp 559 - 589, https://arxiv.org/abs/1505.02900.

Warning. Depending on the authors, t may have to be replaced with 1/t. The Pari/GP convention is the same as the one in Magma, but is the inverse of the one in the last reference.

This formula does not make sense and is not valid for *bad primes* p: a *wild prime* is a prime which divides a denominator of the α_j or β_i. If a prime p is not wild, it can be *good* if $v_p(t) = v_p(t-1) = 0$, or *tame* otherwise. The *local Euler factor* P_p at a good prime p is then given by the usual formula $-\log P_p(T) = \sum_{f \ge 1} (N(H,t; p^f)\, T^f)/(f)$, and in the case of HGM's P_p is always a polynomial (note that the Euler factor used in the global L-function is $1/P_p(p^{-s})$). At a tame prime p it is necessary to modify the above formula, and usually (but not always) the degree of the local Euler factor decreases. Wild primes are currently not implemented by a formula but can be guessed via the global functional equation (see the next section). Continuing the previous example, we find

Notes

This is a faithful port from Magma. I have not used it very much.

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Hypergeometric motives

This is largely a port of the corresponding package in Magma. One important conventional difference: the motivic parameter \(t\) has been replaced with \(1/t\) to match the classical literature on hypergeometric series. (E.g., see [BeukersHeckman])

The computation of Euler factors is currently only supported for primes \(p\) of good or tame reduction.

AUTHORS:

- Frédéric Chapoton
- Kiran S. Kedlaya

EXAMPLES:

Sage Python

```
sage: from sage.modular.hypergeometric_motive import HypergeometricData as Hyp
sage: H = Hyp(cyclotomic=([30], [1,2,3,5]))
sage: H.alpha_beta()
([1/30, 7/30, 11/30, 13/30, 17/30, 19/30, 23/30, 29/30],
 [0, 1/5, 1/3, 2/5, 1/2, 3/5, 2/3, 4/5])
sage: H.M_value() == 30**30 / (15**15 * 10**10 * 6**6)
True
sage: H.euler_factor(2, 7)
T^8 + T^5 + T^3 + 1
```

REFERENCES:

- [BeukersHeckman]
- [Benasque2009]
- [Kat1991]
- [MagmaHGM]
- [Fedorov2015]
- [Roberts2017]
- [Roberts2015]
- [RRV2022]
- [BeCoMe]

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Notes

The computation of Frobenius traces is somewhat less efficient than Magma in the prime case, but more efficient in the prime-power case.

There is not yet much extra functionality (e.g., for computing the approximate functional equation), but this is under active development.

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Contents

Hypergeometric L-functions via amortizedHGM: a snapshot

README | GPL-3.0 license

amortizedHGM

Sage code supporting

- [arXiv DOI] **Hypergeometric L-functions in average polynomial time** by Edgar Costa, Kiran Kedlaya and David Roe
- [arXiv] **Hypergeometric L-functions in average polynomial time, II** by Edgar Costa, Kiran Kedlaya and David Roe

The previous version of the code was not a pip package, and can be found here.

This code depends on pyrforest.

Install

```
sage -pip install --upgrade  git+https://github.com/edgarcosta/amortizedHGM.git@main
```

If you don't have permissions to install it system wide, please add the flag `--user` to install it just for you.

```
sage -pip install --user --upgrade git+https://github.com/edgarcosta/amortizedHGM.git@main
```

Example

```
sage: from amortizedHGM import AmortizingHypergeometricDatamodp # the version of the first paper
....: H = AmortizingHypergeometricDatamodp(10**2,   cyclotomic=([4,2,2], [3,3]))
....: H.amortized_padic_H_values(1/5) # this uses the rforest library
{11: 9,
 13: 9,
 17: 13,
```

Packages

No packages published
Publish your first package

Contributors 4

- **edgarcosta** Edgar Costa
- **kedlaya**
- **roed314** David Roe
- **stevehuang235** Steve Huang

Languages

- **Python** 90.6%
- **Cython** 9.0%
- **Makefile** 0.4%

Notes

The point here is to compute prime Frobenius traces up to some cutoff using **remainder forests**. A useful analogy is computing truncated hypergeometric sums

$$\sum_{m=0}^{\lfloor \gamma p \rfloor - 1} \frac{(\alpha_1)_m \cdots (\alpha_n)_m}{(\beta_1)_m \cdots (\beta_n)_m} z^m \pmod{p} \qquad \text{for all } p \leq X$$

by exploiting the overlaps in the sums to amortize the complexity over p.

As such, there is no attempt to handle bad primes, and even some small good primes get skipped.

It would in principle be useful to implement a similar amortization for p^k-traces for fixed $k > 1$, but this might be too complicated to be useful in practice. Note that for the use case of computing all Dirichlet coefficients up to X, one only needs p^k-traces for $p \leq X^{1/k}$.

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The table below gives for each degree d and weight w shown, the number of corresponding hypergeometric families.

$w \setminus d$	1	3	5	7	9	2	4	6	8	10
0	1	3	7	21	13	3	11	23	51	23
2		10	93	426	1836		30	234	1234	4475
4			47	414	2878			84	894	5737
6				142	1263				204	1936
8					363					426
1						10	74	287	1001	2197
3							47	487	3247	14397
5								142	1450	10260
7									363	3407
9										812

Families above are separated by type – those with odd weight and even degree are symplectic, otherwise they are orthogonal. Boxes are blank for combinations of weight and degree which cannot occur.

Some interesting families of hypergeometric motives, a random family or a random hypergeometric motive.

Learn more
Source and acknowledgments
Completeness of the data
Reliability of the data
Hypergeometric motive label

Search for hypergeometric families

Enter values into one or more boxes to restrict the list of families returned.

Degree	4	e.g. 4	Weight	3	e.g. 3			
Family Hodge vector	[1,1,1,1]	e.g. [1,1,1,1]	A	[3,2,2]	e.g. [3,2,2]	B	[6,4]	e.g. [6,4]
Prime p	2		A_p	[2,2,1,1]	e.g. [2,2,1,1]	B_p	[2,2,1,1]	e.g. [2,2,1,1]
			$A_p^\perp$	[2,2,1,1]	e.g. [2,2,1,1]	$B_p^\perp$	[2,2,1,1]	e.g. [2,2,1,1]

Display: List of families Random family

Search for individual hypergeometric motives

Enter values into the boxes above and the boxes below to obtain the list of corresponding hypergeometric motives.

Notes

This is work in progress: the tables contain hypergeometric families and some data about specializations, but not yet any L-functions. These will eventually be computed using the aforementioned tools. Help wanted!

Contents

That QR code again