Space vectors forming rational angles

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(with Sasha Kolpakov, Bjorn Poonen, and Michael Rubinstein)

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These slides can be downloaded from https://kskedlaya.org/slides/.

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I acknowledge that my workplace occupies unceded ancestral land of the Kumeyaay Nation.

Contents

Statement of the problem

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Find all sets of lines through the origin in $\mathbb{R}^3$ with the property that the angle formed by any two of the lines is a rational multiple of π (or equivalently, a rational number of degrees). We call such a set a **rational-angle line configuration**.

Of course, we consider these sets up to isometries of $\mathbb{R}^3$ fixing the origin (rotation, reflection). Also, it is enough to classify *maximal* sets with this property (i.e., sets to which no additional line can be added).

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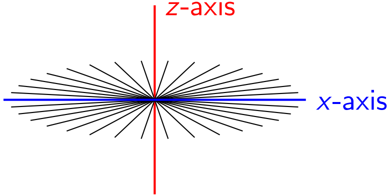

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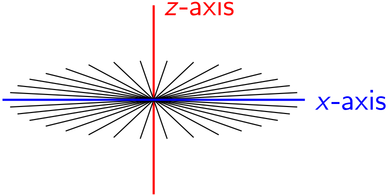

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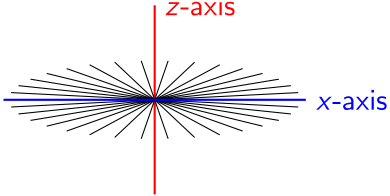

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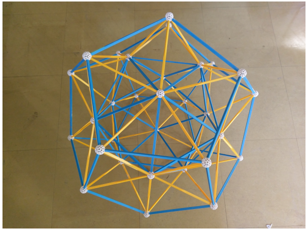
Consider a dodecahedron, and draw the 15 lines from the center to the midpoints of each of the 30 edges. (These are also the midpoints of the edges of an icosahedron, or the vertices of an **icosidodecahedron**.)



source on right: wayfair.com

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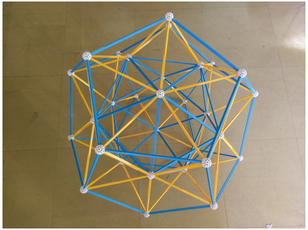

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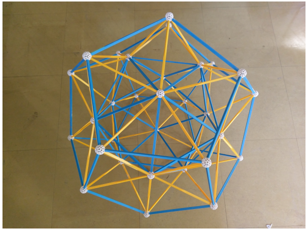

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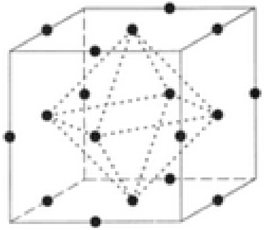
Yet another maximal configuration

Consider a cube with vertices $(\pm 1, \pm 1, \pm 1)$. Draw the lines from the center to each of the midpoints of the edges, and to each of the centers of the faces; there are $(12+6)/2 = 9$ distinct lines in this configuration. (For those familiar with Lie algebras, this is the B_3 root system.)



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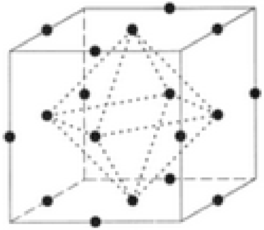

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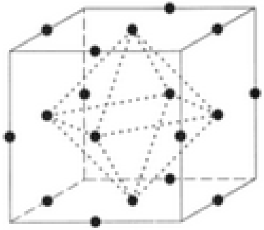

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Example

There are 5 different 8-line configurations consisting of seven central diagonals of a regular 60-gon, plus an eighth line not in the same plane.

Example

There are **infinitely many** 6-line configurations of this form. Take two perpendicular lines L_1 and L_2. Choose a plane containing L_1 but not L_2, and rotate by $\pm\frac{2\pi}{3}$ around the normal to that plane to get four more lines.

Example

There are **infinitely many** 6-line configurations of this form. Take a "fan" of five lines L_1, L_2, L_3, L_4, L_5 spaced by angles of θ. For θ in a suitable range, there is a sixth line perpendicular to L_3, making angles of $\frac{\pi}{3}$ with L_2 and L_4, and angles of θ with L_1 and L_5.

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A classification theorem

Theorem (KKPR, 2020)

The maximal rational-angle line configurations are classified as in the following table.

n	number of maximal rational-angle n-line configurations
$\aleph_0$	1
15	1
9	1
8	5
6	22, plus 5 one-parameter families
5	29, plus 2 one-parameter families
4	228, plus 10 one-parameter families and 2 two-parameter families
3	1 three-parameter family (the trivial one)

Contents

Tetrahedra with rational dihedral angles

Theorem

Up to symmetry, any tetrahedron in $\mathbb{R}^3$ with all dihedral angles rational is either one of 59 sporadic examples (next slide) or has one of the forms

$$\left(\frac{\pi}{2}, \frac{\pi}{2}, \pi - 2x, \frac{\pi}{3}, x, x\right) \qquad \text{for } \frac{\pi}{6} < x < \frac{\pi}{2},$$

$$\left(\frac{5\pi}{6} - x, \frac{\pi}{6} + x, \frac{2\pi}{3} - x, \frac{2\pi}{3} - x, x, x\right) \qquad \text{for } \frac{\pi}{6} < x \leq \frac{\pi}{3}.$$

This answers a question of Conway–Jones from 1976. (Given such a tetrahedron in $\mathbb{R}^3$, pick a point in its interior; the lines through that point perpendicular to the faces form a rational-angle 4-line configuration.)

Sporadic tetrahedra (key on the next slide)

N	$(\alpha_{12}, \alpha_{34}, \alpha_{13}, \alpha_{24}, \alpha_{14}, \alpha_{23})$ as multiples of π/N
12	$(3, 4, 3, 4, 6, 8) = H_2(\pi/4)$
24	(5, 9, 6, 8, 13, 15)
12	$(3, 6, 4, 6, 4, 6) = T_0$
24	(7, 11, 7, 13, 8, 12)
15	$(3, 3, 3, 5, 10, 10) = T_{18}$, (2, 4, 4, 4, 10, 10), (3, 3, 4, 4, 9, 11)
15	$(3, 3, 5, 5, 9, 9) = T_7$
15	$(5, 5, 5, 9, 6, 6) = T_{23}$, (3, 7, 6, 6, 7, 7), (4, 8, 5, 5, 7, 7)
21	(3, 9, 7, 7, 12, 12), (4, 10, 6, 6, 12, 12), (6, 6, 7, 7, 9, 15)
30	$(6, 12, 10, 15, 10, 20) = T_{17}$, (4, 14, 10, 15, 12, 18)
60	(8, 28, 19, 31, 25, 35), (12, 24, 15, 35, 25, 35), (13, 23, 15, 35, 24, 36), (13, 23, 19, 31, 20, 40)
30	$(6, 18, 10, 10, 15, 15) = T_{13}$, (4, 16, 12, 12, 15, 15), (9, 21, 10, 10, 12, 12)
30	$(6, 6, 10, 12, 15, 20) = T_{16}$, (5, 7, 11, 11, 15, 20)
60	(7, 17, 20, 24, 35, 35), (7, 17, 22, 22, 33, 37), (10, 14, 17, 27, 35, 35), (12, 12, 17, 27, 33, 37)
30	$(6, 10, 10, 15, 12, 18) = T_{21}$, (5, 11, 10, 15, 13, 17)
60	(10, 22, 21, 29, 25, 35), (11, 21, 19, 31, 26, 34), (11, 21, 21, 29, 24, 36), (12, 20, 19, 31, 25, 35)
30	$(6, 10, 6, 10, 15, 24) = T_6$
60	(7, 25, 12, 20, 35, 43)
30	$(6, 12, 6, 12, 15, 20) = T_2$
60	(12, 24, 13, 23, 29, 41)
30	$(6, 12, 10, 10, 15, 18) = T_3$, (7, 13, 9, 9, 15, 18)
60	(12, 24, 17, 23, 33, 33), (14, 26, 15, 21, 33, 33), (15, 21, 20, 20, 27, 39), (17, 23, 18, 18, 27, 39)
30	$(6, 15, 6, 18, 10, 20) = T_4$, (6, 15, 7, 17, 9, 21)
60	(9, 33, 14, 34, 21, 39), (9, 33, 15, 33, 20, 40), (11, 31, 12, 36, 21, 39), (11, 31, 15, 33, 18, 42)
30	$(6, 15, 10, 15, 12, 15) = T_1$, (6, 15, 11, 14, 11, 16), (8, 13, 8, 17, 12, 15),
	(8, 13, 9, 18, 11, 14), (8, 17, 9, 12, 11, 16), (9, 12, 9, 18, 10, 15)
30	$(10, 12, 10, 12, 15, 12) = T_5$
60	(19, 25, 20, 24, 29, 25)

How to read the table

Each tetrahedron is represented by an integer N and a list of six integers, representing the dihedral angles $\alpha_{12}, \alpha_{34}, \alpha_{13}, \alpha_{24}, \alpha_{14}, \alpha_{23}$ as multiples of $\frac{\pi}{N}$. (Here α_{ij} means the angle between faces i and j.)

The extra labels indicate examples of tetrahedra that we found in the literature as examples of rectifiable tetrahedra. All of these come from 4-line configurations within the maximal 9-line and 15-line configurations.

The groups between horizontal lines are orbits for a certain "extra" symmetry group (more on this shortly).

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Rectifiable tetrahedra

In $\mathbb{R}^2$, any two polygons with equal area are **scissors-congruent**: one can be cut up into finitely many pieces and reassembled to form the other.

Source: Wikimedia Commons

This fails in $\mathbb{R}^3$! In 1901, Dehn constructed a numerical invariant of scissors-congruence, which is zero for a cube and nonzero for a regular tetrahedron. (See Numberphile for the definition.)

Around 1960, Sydler showed* that Dehn's invariant is complete: any two polyhedra with the same volume and Dehn invariant are scissors-congruent. As a corollary, any tetrahedron with Dehn invariant zero is scissors-congruent to a cube (a/k/a **rectifiable**).

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More motivation: in 1980, Debrunner showed that any tetrahedron that tiles† $\mathbb{R}^3$ is rectifiable.‡ This gives a new proof that one cannot tile $\mathbb{R}^3$ with regular tetrahedra (as falsely claimed by Aristotle).

I have no idea how to classify general rectifiable tetrahedra, or even to prove any sort of finiteness statement about them (allowing some parametric families). However, one can do the latter for some intermediate subclasses; more on this later.

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Theorem (Ponzano–Regge)

For any tetrahedron with edge lengths $(\ell_{12}, \ell_{34}, \ell_{13}, \ell_{24}, \ell_{14}, \ell_{23})$ *and dihedral angles* $(\alpha_{12}, \alpha_{34}, \alpha_{13}, \alpha_{24}, \alpha_{14}, \alpha_{23})$*, there is another with edges*

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Consequences of Regge symmetry

The family of tetrahedra with dihedral angles $\left(\frac{\pi}{2}, \frac{\pi}{2}, \pi - 2x, \frac{\pi}{3}, x, x\right)$ was discovered by Hill in 1895. Applying a Regge symmetry gives the family $\left(\frac{5\pi}{6} - x, \frac{\pi}{6} + x, \frac{2\pi}{3} - x, \frac{2\pi}{3} - x, x, x\right)$ found by Poonen–Rubinstein in the 1990s (without knowing about Regge symmetry).

Together with the action of S_4 on faces, the Regge symmetry generates the larger group $W(D_6)$ of order 23040 acting on isomorphism classes of labeled tetrahedra. Our table of sporadic tetrahedra indicates orbits for this larger group.

In particular, all but three sporadic tetrahedra are "explained" by the classical examples (coming from the 9-line and 15-line configurations) via this larger symmetry group.

The Regge symmetry also applies to degenerate tetrahedra; that is, it acts on rational-angle configurations of 4 lines. We can thus exploit it in the proof of both theorems.

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The family of tetrahedra with dihedral angles $\left(\frac{\pi}{2}, \frac{\pi}{2}, \pi - 2x, \frac{\pi}{3}, x, x\right)$ was discovered by Hill in 1895. Applying a Regge symmetry gives the family $\left(\frac{5\pi}{6} - x, \frac{\pi}{6} + x, \frac{2\pi}{3} - x, \frac{2\pi}{3} - x, x, x\right)$ found by Poonen–Rubinstein in the 1990s (without knowing about Regge symmetry).

Together with the action of S_4 on faces, the Regge symmetry generates the larger group $W(D_6)$ of order 23040 acting on isomorphism classes of labeled tetrahedra. Our table of sporadic tetrahedra indicates orbits for this larger group.

In particular, all but three sporadic tetrahedra are "explained" by the classical examples (coming from the 9-line and 15-line configurations) via this larger symmetry group.

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Contents

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Some subtleties with (this) computer-assisted proof

Some of the issues that arise in this particular computer-assisted proof:

- Since our programs are part of the proof, they need to be documented and made publicly available. We use Jupyter notebooks to present some of our code.
- Some of the code (written in C) uses floating-point arithmetic. In order to make this rigorous, we must pay some attention to the possible roundoff errors that can occur.
- Some of the code (written in Sage) depended on features that we had to implement ourselves. We ended up submitting some code to the Sage project with these features (plus bugfixes).
- Some of the code is written in Magma, which is not an open-source system. This code needs to be documented especially well so that it can be checked by porting over to another language.
- The code needs to be written with some care, so that it terminates in a reasonable amount of time! (Say, less than a week on a single CPU.)

Finding 4-line configurations

The main difficulty is to classify rational-angle 4-line configurations. To find larger ones, we start with each possible set of 4, then repeatedly try to extend it so that every 4-element subset of the result is in the original list.[§]

To find 4-line configurations, we first classify 6-tuples of angles $(\theta_{ij})_{1\leq i<j\leq 4}$ that satisfy the following condition:

$$\det \begin{pmatrix} 1 & \cos\theta_{12} & \cos\theta_{13} & \cos\theta_{14} \\ \cos\theta_{12} & 1 & \cos\theta_{23} & \cos\theta_{24} \\ \cos\theta_{13} & \cos\theta_{23} & 1 & \cos\theta_{34} \\ \cos\theta_{14} & \cos\theta_{24} & \cos\theta_{34} & 1 \end{pmatrix} = 0.$$

Proof that this condition is necessary: choose unit vectors along the lines $L_1, \dots, L_4$ and make the 3×4 matrix A with those vectors as the columns. Then A has rank at most 3 and $A^T A$ is the matrix displayed above.

[§]This is somewhat complicated by the presence of parametric families of 4-line configurations, which means we must work in a manner that allows specialization.

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An algebraic translation

For $z_{jk} = e^{i\theta_{jk}}$, the algebraic condition we wrote down becomes

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This is a Laurent polynomial in the six variables z_{jk}, which we want to solve in roots of unity. This is a class of problem with applications to many branches of mathematics (Euclidean and non-Euclidean geometry, finite group theory, knot theory, operator algebras, graph theory, dynamical systems...).

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An approach via cyclotomic relations

One way to solve such problems is to classify all the ways that a sum of a fixed number of roots of unity can equal 0. For example, if $\zeta_1, \ldots, \zeta_6$ are six roots of unity that sum to zero, then either:

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(Exercise: prove this!)

This classification is known for at most 21 roots of unity (work of Mann, Włodarski, Conway–Jones, Poonen–Rubinstein, Christie–Dykema–Klep). However, our determinant is a sum of 105 monomials, so this isn't immediately helpful.

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For example, given a Laurent polynomial $f(x, y)$ over $\mathbb{Q}$, any solution of $f(x, y) = 0$ in roots of unity is also a solution of one of the polynomials

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In practice, this approach works very well in 2 variables, barely in 3 variables, and not at all in 4 or more variables.

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We introduce a third approach: classify relations among roots of unity modulo 2 (in the ring of algebraic integers). For example, if $\zeta_1, \ldots, \zeta_6$ are six roots of unity that sum to zero mod 2, then either:

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This helps because our determinant reduces mod 2 to a Laurent polynomial with only 12 monomials:

$$z_{12}^2 z_{34}^2 + z_{12}^{-2} z_{34}^{-2} + z_{12}^2 z_{34}^{-2} + z_{12}^{-2} z_{34}^2 + \cdots$$

and this is in the range we can handle (following Poonen–Rubinstein).

Aside: With D'Nelly–Warady, we showed that all mod 2 relations of length ≤ 18 lift to genuine relations of the same length. This is best possible.

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The strategy

We now classify rational-angle 4-line configurations as follows.

- Do a C computation to find angle solutions with small denominator (up to 420), discarding those in known parametric families. This finds a putative classification **and** provides a key step in the proof.
- Write down all relations among the 12 monomials that persist mod 2.
- For each relation, make a system of equations that imposes these relations plus the vanishing of the original determinant.
- Use Regge symmetries to reduce the number of systems (down to a few hundred).
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- Use Regge symmetries to reduce the number of systems (down to a few hundred).
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We now classify rational-angle 4-line configurations as follows.

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More on rectifiable tetrahedra

For a rectifiable tetrahedron, the $\mathbb{Q}$-span of the dihedral angles in $\mathbb{R}/\mathbb{Q}\pi$ has dimension at most 5. When the dimension is 0, we have a tetrahedron with rational dihedral angles. (This dimension is again preserved by Regge symmetry.)

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It is unclear what happens for other values of the span dimension. The Hill family generically gives span dimension 1; Hill found two other families with generic span dimension 2, and Chentouf–Sun identified a third (distinct even up to Regge symmetry).

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There are known examples of simplices in $\mathbb{R}^n$ with rational dihedral angles for all $n \geq 4$ (Maehara–Martini). Is it feasible to classify them for $n = 4$?

Using our techniques, probably not: the analogue of the determinant equation is a polynomial in 10 variables with 604 monomials, with no useful structure mod 2.

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