The Honda-Tate problem for K3 surfaces

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These slides can be downloaded from https://kskedlaya.org/slides/.

Arithmetic geometry of K3 surfaces
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I acknowledge that my workplace occupies unceded ancestral land of the Kumeyaay Nation.

Contents

Zeta functions of algebraic varieties

Let X be a smooth proper $\mathbb{F}_q$-scheme for some finite field $\mathbb{F}_q$ (with $q = p^f$ for some prime p). The **zeta function** of X is the rational function represented by the power series

$$Z_X(T) = \prod_{x \in X^\circ} (1 - T^{\deg_{\mathbb{F}_q}(x)})^{-1} = \exp\left(\sum_{n=1}^{\infty} \frac{T^n}{n} \#X(\mathbb{F}_{q^n})\right)$$

where X° denotes the set of closed points of X. We have the factorization

$$Z_X(T) = \prod_{i=0}^{2\dim(X)} L_{X,i}(T)^{(-1)^i} = \frac{L_{X,1}(T) \cdots L_{X,2\dim(X)-1}(T)}{L_{X,0}(T) \cdots L_{X,2\dim(X)}(T)}$$

where $L_{X,i}(T) \in 1 + T\mathbb{Z}[T]$ is a polynomial with all $\mathbb{C}$-roots on the circle $|T| = q^{-i/2}$.

Abelian varieties and the Honda–Tate theorem

For $X = A$ an abelian variety, we have $L_{A,i}(T) = \wedge^i L_{A,1}(T)$; that is, the roots of $L_{A,i}(T)$ are the products of i-element subsets of roots of $L_{A,1}$.

Theorem (Tate)

For A, A' two abelian varieties over $\mathbb{F}_q$, $L_{A,1}(T) = L_{A',1}(T)$ iff A' is isogenous to A.

Theorem (Honda)

For every irreducible polynomial $P(T) \in 1 + T\mathbb{Z}[T]$ with all $\mathbb{C}$-roots on the circle $|T| = q^{-1/2}$, there exists a simple abelian variety A such that $L_{A,1}(T) = P(T)^e$ for some (explicit) positive integer e.

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Zeta functions of curves

For $X = C$ a (geometrically irreducible) curve of genus g, we have

$$L_{C,0}(T) = 1 - T, \qquad L_{C,2}(T) = 1 - qT, \qquad \deg(L_{C,1}) = 2g.$$

For each positive integer n we also have

$$0 \leq \#\{x \in X^\circ \colon \deg_{\mathbb{F}_q}(x) = n\} = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) \#X(\mathbb{F}_{q^d}).$$

Sample corollary: $\limsup_{g \to \infty} \frac{\#X(\mathbb{F}_q)}{g} \leq \sqrt{q} - 1$ (Ihara, Drinfeld–Vlăduț).

For $g \geq 4$, it is difficult to identify those C for which $L_{C,1}(T)$ takes a fixed value. E.g., to classify double covers $C' \to C$ with trivial relative class group, one must find all genus 7 curves with certain values of $L_{C,1}(T)$; this requires an exhaust using Mukai's description of canonical curves in genus 7.

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A question

Vlăduț–Nogin–Tsfasman (2018)[1]: The question of what zeta functions are realized for K3 surfaces over a given finite field remains open.

In this talk, we will:

- record the known constraints on $L_X(T)$;
- report some theoretical and computational existence results;
- introduce the related question for cubic fourfolds and survey some results;
- introduce the related question for noncommutative K3 surfaces.

[1] Вопрос о том, какие дзета-функции реализуются для поверхностей типа K3 над данным конечным полем, остается открытым.

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Status of the Tate conjecture for K3 surfaces

The Tate conjecture is known:

- for a K3 surface (Artin–Swinnerton-Dyer, Rudakov–Shafarevich–Zink, Nygaard–Ogus, Maulik, Charles, Madapusi Pera, Kim–Madapusi Pera, K. Ito–T. Ito–Koshikawa);
- for the square of a K3 surface (Ito–Ito–Koshikawa, Yang);
- for the product of two K3 surfaces of distinct heights (Zarhin, Yu–Yui) but not in general.

Consequently, it is not clear whether K3 surfaces with the same zeta function are isogenous (i.e., there is a correspondence between them which induces an isomorphism on cohomology).

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We have $L_{X,2}(T) = q^{-1} L_X(qT)$ where (by "Newton above Hodge")

$$L_X(T) = q + c_1 T + \cdots + c_{22} T^{22}, \qquad c_i \in \mathbb{Z}.$$

The polyonmial $L_X(T)$ has all $\mathbb{C}$-roots on the unit circle. Moreover, $L_X(T)$ is a palindromic polynomial times either $(1-T)^2$ or $(1-T)(1+T)$.

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The Néron–Severi group

Let $\overline{\rho}$ denote the Picard rank of $X_{\overline{\mathbb{F}}_q}$. Then $L_X(T) = L_{X,\mathrm{alg}}(T) L_{X,\mathrm{trans}}(T)$ where:

- $L_{X,\mathrm{alg}}(T)$ is a product of cyclotomic polynomials of degree $\overline{\rho}$ (which must thus be even);
- $L_{X,\mathrm{trans}}(T)$ is palindromic with leading coefficient q.

One possibility allowed here is that $\overline{\rho} = 22$. In this case, X is **supersingular** (and $L_{X,\mathrm{trans}}(T)$ is the constant q). One can further classify supersingular K3 surfaces according to their **Artin invariant**, but the zeta function is insensitive to this.

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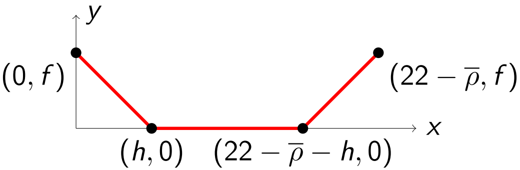
The Newton polygon

For X not supersingular, $L_{X,\mathrm{trans}}(T)$ has this Newton polygon for some $h \in \{1, \dots, \frac{22-\overline{\rho}}{2}\}$:



That is, $c_h \not\equiv 0 \pmod{p}$ and $c_i \equiv 0 \pmod{p^{\lceil f(1-i/h) \rceil}}$ for $i = 1, \dots, h-1$. Moreover, $L_{X,\mathrm{trans}}(T) = Q(T)^e$ for some irreducible $Q(T) \in \mathbb{Z}[T]$ and some e dividing $\gcd(f, h, \frac{22-\overline{\rho}}{2})$ (Yu–Yui), and $Q(T)$ has a unique irreducible factor in $\mathbb{Q}_p[T]$ with negative slope (Taelman).

The quantity h coincides with the **Artin–Mazur height** of the formal Brauer group of X, and with the **quasi-F-split height** of X in the sense of Kawakami–Takamatsu–Yoshikawa.

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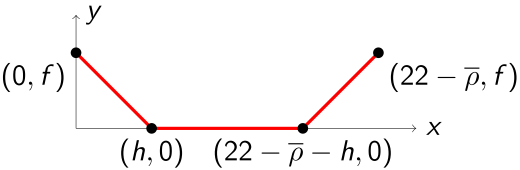

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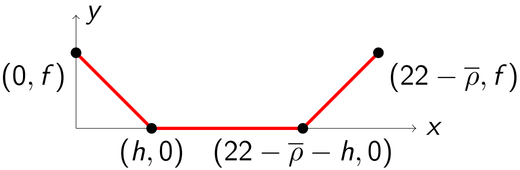

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An example of $e > 1$

Take $q = p^2$ (with $p > 2$), so that there exists a supersingular elliptic curve E_1 over $\mathbb{F}_q$ with $L_{E_1,1}(T) = (1 - pT)^2$. Let E_2 be an ordinary elliptic curve over $\mathbb{F}_q$ and write $L_{E_2,1}(T) = (1 - \alpha T)(1 - \overline{\alpha} T)$. Let X be the Kummer of $E_1 \times E_2$; then

$$L_X(T) = Q(T)^2, \qquad Q(T) := (\alpha - \overline{\alpha} T)(\overline{\alpha} - \alpha T).$$

The Artin–Tate formula

Write $L_X(T) = (1 - T)^\rho P(T)$ with $\rho = \operatorname{rank} \mathrm{NS}(X)$. Then

$$P(1) = |\Delta| \# \mathrm{Br}(X)$$

where Δ is the discriminant of $\mathrm{NS}(X)$ and $\mathrm{Br}(X)$ is the Brauer group, which is finite with order a perfect square.

Elsenhans–Jahnel observed that comparing this condition for X and $X_{\mathbb{F}_{q^2}}$ shows that $P(-1)$ is always a perfect square (which may be 0).

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Nonnegativity

For each positive integer n we also have

$$0 \leq \#\{x \in X^\circ \colon \deg_{\mathbb{F}_q}(x) = n\} = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) \#X(\mathbb{F}_{q^d}).$$

- For $q \geq 23$ this condition is empty.
- For $5 \leq q \leq 19$ it is equivalent to $\#X(\mathbb{F}_q) \geq 0$.
- For $3 \leq q \leq 4$ it also includes $\#X(\mathbb{F}_{q^2}) \geq \#X(\mathbb{F}_q)$.
- For $q = 2$ it also includes $\#X(\mathbb{F}_{q^3}) \geq \#X(\mathbb{F}_q)$ and $\#X(\mathbb{F}_{q^4}) \geq \#X(\mathbb{F}_{q^2})$.

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- For $q = 2$ it also includes $\#X(\mathbb{F}_{q^3}) \geq \#X(\mathbb{F}_q)$ and $\#X(\mathbb{F}_{q^4}) \geq \#X(\mathbb{F}_{q^2})$.

Nonnegativity

For each positive integer n we also have

$$0 \leq \#\{x \in X^\circ \colon \deg_{\mathbb{F}_q}(x) = n\} = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) \#X(\mathbb{F}_{q^d}).$$

- For $q \geq 23$ this condition is empty.
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Enumerating candidates

It is straightforward to enumerate candidates for $L_{X,\mathrm{alg}}(T)$, as this does not depend on q.

To enumerate candidates for $L_{X,\mathrm{trans}}(T)$, one can use code for finding Weil polynomials in SageMath. Sample results (which required several days on a laptop):

- For $q = 2$ one gets 1672565 candidates for $L_X(T)$.
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To go further would require more theoretical (and perhaps computational) optimization. E.g., if $L_{X,\mathrm{trans}}(T) = q + aT + \cdots$, one can improve the trivial upper bound

$$|a| \leq q \deg(L_{X,\mathrm{trans}}) \qquad \text{to} \qquad |a| \leq q \deg(L_{X,\mathrm{trans}}) \log 2$$

by writing $L_{X,\mathrm{trans}}(\pm 1) = q \prod_\alpha (2 \pm (\alpha + \overline{\alpha}))$ and using that $-x - \log(1-x) \geq 0$ on $[-1, 1]$. (One can continue by evaluating at other roots of unity.)

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Which polynomials occur as $L_X(T)$ for some supersingular K3 surface X over $\mathbb{F}_q$?

This amounts to asking for a classification of the Frobenius action on $\mathrm{NS}(X_{\overline{\mathbb{F}}_q}) \otimes \mathbb{Q}$; note that the candidates form a finite set which is independent of q. Some results on the Frobenius trace have been obtained by Rybakov using generalized Kummer surfaces.

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For every p, does there exist a K3 surface over $\mathbb{F}_p$ of height h for each $h \in \{1, \ldots, 10\}$?

This was confirmed for $p = 2$ by K–Sutherland by a census (see below); for $p = 3$ by Kawakami–Takamatsu–Yoshikawa using a Fedder-type criterion for quasi-F-split height; and for $p = 5, 7$ by Batubara–Garzella–Pan using a different algorithm for the Fedder-type criterion plus GPU computations.

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Let $P(T)$ be a polynomial that "looks like $L_{X,\mathrm{trans}}(T)$"; that is,

- $P(T) \in q + T\mathbb{Z}[T]$ has degree in $\{2, 4, \dots, 20\}$;
- every $\mathbb{C}$-root of $P(T)$ has norm 1 but is not a root of unity;
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For d a positive integer, let $P^{(d)}(T) \in q^d + T\mathbb{Z}[T]$ be the polynomial obtained from $P(T)$ by raising each root to the d-th power.

Theorem (Taelman, K. Ito)

Assume[a] $p \geq 7$. Then for any $P(T)$ as above, there exist a positive integer d and a K3 surface X over $\mathbb{F}_{q^d}$ such that $L_{X,\mathrm{trans}}(T) = P^{(d)}(T)$.

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A Honda–Tate conjecture for K3 surfaces

Conjecture

For any $P(T)$ as on the previous slide, there exists a K3 surface X over $\mathbb{F}_q$ such that $L_{X,\mathrm{trans}}(T) = P(T)$ (with no base extension required).

To interrogate this conjecture, one can ask whether there is a choice of $L_{X,\mathrm{alg}}(T)$ depending on $P(T)$ so that the resulting $L_X(T)$ satisfies the nonnegativity and Elsenhans–Jahnel constraints.

- For $q \geq 4, \deg(P) < 20$, one may take $L_{X,\mathrm{alg}}(T) = (1+T)(1-T)^{21-\deg(P)}$. The E–J condition is vacuous.
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- For $q = 2, 3$, one of $L_{X,\mathrm{alg}}(T) = (1 \pm T)(1-T)^{21-\deg(P)}$ always works.
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An example via Kummer surfaces

Suppose $\deg(P) = 4$ and write $P(T) = (1 - \beta_1 T)(1 - \beta_2 T)(1 - \overline{\beta}_1 T)(1 - \overline{\beta}_2 T)$. Set

$$\alpha_1 := q^{-1}\beta_1\beta_2, \qquad \alpha_2 := q^{-1}\beta_1\overline{\beta}_2.$$

There exists an abelian surface A over $\mathbb{F}_{q^2}$ with

$$L_{A,1}(T) = (1 - \alpha_1 T)(1 - \alpha_2 T)(1 - \overline{\alpha}_1 T)(1 - \overline{\alpha}_2 T).$$

Its Kummer $X \sim A/(-1)$ then satisfies $L_X(T) = P^{(2)}(T)$. Can we descend X to $\mathbb{F}_q$?

Idea: let B be the Weil restriction of A to $\mathbb{F}_q$. Define the polynomial

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Suppose $\deg(P) = 4$ and write $P(T) = (1 - \beta_1 T)(1 - \beta_2 T)(1 - \overline{\beta}_1 T)(1 - \overline{\beta}_2 T)$. Set

$$\alpha_1 := q^{-1}\beta_1\beta_2, \qquad \alpha_2 := q^{-1}\beta_1\overline{\beta}_2.$$

There exists an abelian surface A over $\mathbb{F}_{q^2}$ with

$$L_{A,1}(T) = (1 - \alpha_1 T)(1 - \alpha_2 T)(1 - \overline{\alpha}_1 T)(1 - \overline{\alpha}_2 T).$$

Its Kummer $X \sim A/(-1)$ then satisfies $L_X(T) = P^{(2)}(T)$. Can we descend X to $\mathbb{F}_q$?

Idea: let B be the Weil restriction of A to $\mathbb{F}_q$. Define the polynomial

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Censuses of smooth quartic surfaces

K–Sutherland compiled a census of smooth quartic surfaces over $\mathbb{F}_2$, and computed their zeta functions using optimized point enumeration. This gives only 52755 zeta functions, compared to the 1672565 candidates for $L_X(T)$.

However, this includes all of the 1995 values of $L_X(T)$ for which

$$L_{X,\mathrm{alg}}(T) = (1+T)(1-T), \qquad L_{X,\mathrm{trans}}(1) = 2, \qquad L_{X,\mathrm{trans}}(-1) > 2;$$

note that these conditions (almost) force degree 4 by the Artin–Tate formula.

A similar census of smooth quartic surfaces over $\mathbb{F}_3$ was made by Costa–Harvey–K. Computation of the corresponding zeta functions using p-adic cohomological methods remains in progress (Garzella–Huang).

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and $L_{X,4}(T) = (1 - q^2T)q^{-1}L_X(q^2T)$ where $L_X(T) \in q + T\mathbb{Z}[T]$ has degree 22.

Let $\overline{\rho}$ be the Picard rank of $X_{\overline{\mathbb{F}}_q}$. We may again write $L_X(T) = L_{X,\mathrm{alg}}(T)L_{X,\mathrm{trans}}(T)$ where $L_{X,\mathrm{alg}}(T)$ is a product of cyclotomic polynomials of degree[3] $\overline{\rho} - 1$ and $L_{X,\mathrm{trans}}(T)$ is reciprocal with leading coefficient q.

The Newton polygon of $L_{X,\mathrm{trans}}(T)$ obeys the same restrictions as before except that now we must allow $\deg(L_{X,\mathrm{trans}}(T)) = 22$ (and $h = 11$).

There are still nonnegativity constraints on $L_X(T)$ for $q = 2, 3, 4$, but not on $L_{X,\mathrm{trans}}(T)$.

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It is unclear whether one can do the same over $\mathbb{F}_3$. Motivation to do so: Debarre–Laface–Roulleau showed[4] that for $q \neq 3$, every cubic fourfold over $\mathbb{F}_q$ contains an $\mathbb{F}_q$-rational line. That is, for $F(X)$ the Fano variety of lines on a cubic fourfold X over $\mathbb{F}_q$, if $q \neq 3$ then $\#F(X)(\mathbb{F}_q) > 0$.

The Fano variety $F(X)$ is an example of a **hyperkähler** variety (as is a K3 surface). It would be natural to consider the Honda–Tate problem in this context...

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Relationships between K3 surfaces and cubic fourfolds

Nothing prevents $L_X(T)$ from having no cyclotomic factors (or even being irreducible). However, if $1 - T$ divides $L_X(T)$, then $L_X(T)/(1 - T)$ "looks like" it could come from a K3 surface.

In fact there exist several constructions relating K3 surfaces of particular degrees to cubic fourfolds, at least over $\mathbb{C}$ (Hassett et al.). This gives rise to some **Noether–Lefschetz divisors** on the moduli space of cubic fourfolds, characterized by the existence of an extra Hodge class.

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An example of computing zeta functions

Ranestad–Voisin showed that cubic fourfolds geometrically of the form $\sum_{i=1}^{10} a_i^3$, in which each a_i is a linear form and some four are linearly dependent, form a divisor on the moduli space. They conjectured that this is **not** a Noether–Lefschetz divisor.

Costa–Harvey–K. used p-adic cohomological methods to compute that for the cubic fourfold

$$X: x_0^3 + x_1^3 + x_2^3 + (x_0 + x_1 + 2x_2)^3 + x_3^3 + x_4^3 + x_5^3 + 2(x_0 + x_3)^3 + 3(x_1 + x_4)^3 + (x_2 + x_5)^3$$

over $\mathbb{F}_p$ with $p = 127$, $L_X(T)$ is the irreducible degree-22 reciprocal polynomial

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Consequently, X has geometric Picard rank 1, so the Ranestad–Voisin divisor cannot be a Noether–Lefschetz divisor. (This complements work of Addington–Auel who had found rank-1 examples over $\mathbb{F}_2$ in three other divisors found by Ranestad–Voisin.)

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For X a smooth proper $\mathbb{F}_q$-scheme, let $D^b(X)$ denote the bounded derived category of coherent sheaves on X. We consider $D^b(X)$ as an object of a category in which a morphism $D^b(X) \to D^b(Y)$ is a natural isomorphism class of functors of the form

$$C \mapsto R\pi_{2*}(L\pi_1^* C \otimes^L K)$$

for some object $K \in D^b(X \times Y)$ (i.e., a **Fourier–Mukai transform**). When one of these morphisms admits an inverse, we say X and Y are **Fourier–Mukai equivalent** (or **derived equivalent**).

Example (Mukai): for $X = A$ an abelian variety and $K = P$ the Poincaré bundle on $A \times A^\vee$, the Fourier–Mukai transform $D^b(A) \to D^b(A^\vee)$ is an anti-involution. By contrast, if X has ample canonical or anticanonical bundle (e.g., a cubic fourfold) and Y is Fourier–Mukai equivalent to X, then X and Y must be isomorphic (Bondal–Orlov).

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Derived equivalence and zeta functions

A derived equivalence preserves the **even/odd Mukai–Hodge structures**

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When X is a K3 surface, $L_X(T)$ can be recovered from the Frobenius action on the even Mukai–Hodge structure via the Lefschetz trace formula. Consequently, $L_X(T)$ is preserved under derived equivalence (it is unclear whether this holds in higher dimensions).

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Noncommutative K3 surfaces

By analogy with the definition of motives, we may formally enlarge the category of the $D^b(X)$ by adjoining images of idempotent morphisms. One source of such morphisms is **semiorthogonal decompositions**.

Key example (Kuznetsov): for X a smooth cubic fourfold, let $\mathcal{C}$ be the full triangulated subcategory of $F \in D^b(X)$ such that $\mathrm{Hom}(\mathcal{O}_X(i), F[*]) = 0$ for $i = 0, 1, 2$. In common with the derived category of a K3 surface (or a **twisted K3 surface**), $\mathcal{C}$ has the feature that for all $E, F \in \mathcal{C}$, we have functorially

$$\mathrm{Hom}(E, F) \cong \mathrm{Hom}(F, E[2])^\vee.$$

Following Kontsevich (?), we refer to the resulting category as a **noncommutative K3 surface**. This terminology is partly justified by comparing the resulting Hochschild cohomology with that of a K3.

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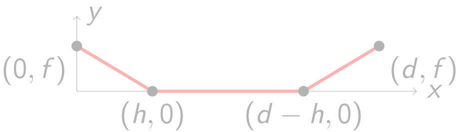
Zeta functions of noncommutative K3 surfaces

Auel–Petok introduced the **zeta function** $Z_{\mathcal{C}}(T)$ of a noncommutative K3 surface $\mathcal{C}$ over $\mathbb{F}_q$. For X the underlying cubic fourfold,

$$Z_{\mathcal{C}}(T) = \frac{q}{(1-T)L_{\mathcal{C}}(qT)(1-q^2T)}, \qquad L_{\mathcal{C}}(T) := L_X(T);$$

i.e.,, the **primitive** middle cohomology of X is a Tate twist of "middle cohomology" of $\mathcal{C}$.

Here $L_{\mathcal{C}}(T) \in q + T\mathbb{Z}[T]$ has degree 22 with $\mathbb{C}$-roots on the unit circle. Write $L_{\mathcal{C}}(T) = L_{\mathcal{C},\text{alg}}(T)L_{\mathcal{C},\text{trans}}(T)$ with $L_{\mathcal{C},\text{alg}}(T)$ the product of the cyclotomic factors; if not constant, $L_{\mathcal{C},\text{trans}}(T)$ is a power of an irreducible $Q(T)$ with Newton polygon:



$$d = \deg(L_{\mathcal{C},\text{trans}}(T)) \in \{2, 4, \dots, 22\}$$
$$h \in \{1, 2, \dots, \tfrac{d}{2}\}.$$

Moreover, in $\mathbb{Q}_p[T]$, $Q(T)$ has a unique irreducible factor with negative slope.

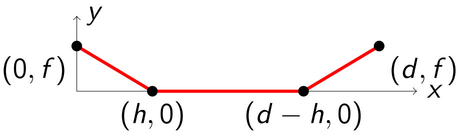
Zeta functions of noncommutative K3 surfaces

Auel–Petok introduced the **zeta function** $Z_{\mathcal{C}}(T)$ of a noncommutative K3 surface $\mathcal{C}$ over $\mathbb{F}_q$. For X the underlying cubic fourfold,

$$Z_{\mathcal{C}}(T) = \frac{q}{(1-T)L_{\mathcal{C}}(qT)(1-q^2T)}, \qquad L_{\mathcal{C}}(T) := L_X(T);$$

i.e.,, the **primitive** middle cohomology of X is a Tate twist of "middle cohomology" of $\mathcal{C}$.

Here $L_{\mathcal{C}}(T) \in q + T\mathbb{Z}[T]$ has degree 22 with $\mathbb{C}$-roots on the unit circle. Write $L_{\mathcal{C}}(T) = L_{\mathcal{C},\mathrm{alg}}(T)L_{\mathcal{C},\mathrm{trans}}(T)$ with $L_{\mathcal{C},\mathrm{alg}}(T)$ the product of the cyclotomic factors; if not constant, $L_{\mathcal{C},\mathrm{trans}}(T)$ is a power of an irreducible $Q(T)$ with Newton polygon:



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Moreover, in $\mathbb{Q}_p[T]$, $Q(T)$ has a unique irreducible factor with negative slope.

The Honda–Tate problem for noncommutative K3 surfaces

Problem (Auel–Petok)

Does every polynomial that "looks like $L_{\mathcal{C}}(T)$" occur as $L_{\mathcal{C}}(T)$ for some noncommutative K3 surface $\mathcal{C}$ over $\mathbb{F}_q$?

Crucially, now we are asking about $L_{\mathcal{C}}(T)$ rather than $L_{\mathcal{C},\mathrm{trans}}(T)$. Again, it is natural to first try to prove this up to an uncontrolled base extension.

Problem (Auel–Petok)

Is there an analogue of the Tate conjecture for noncommutative K3 surfaces over $\mathbb{F}_q$?

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