18.785: Analytic Number Theory, MIT, spring 2007 (K.S. Kedlaya)
Dirichlet characters and Dirichlet L-series

In this unit, we introduce some special multiplicative functions, the Dirichlet characters, and study their corresponding Dirichlet series. We will use these in a subsequent unit to prove Dirichlet's theorem on primes in arithmetic progressions, and the prime number theorem in arithmetic progressions.

1 Dirichlet characters

For N a positive integer, a *Dirichlet character of level* N is an arithmetic function χ which factors through a homomorphism $(\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}$ on integers $n \in \mathbb{N}$ coprime to N, and is zero on integers not coprime to N; such a function is completely multiplicative. Note that the nonzero values must all be N-th roots of unity, and that the characters of level N form a group under termwise multiplication.

For each level N, there is a Dirichlet character taking the value 1 at all n coprime to N; it is called the *principal* (or *trivial*) *character of level* N. A non-principal Dirichlet character of level N is given by the Legendre-Jacobi symbol

$$\chi(n) = \left(\frac{n}{N}\right).$$

Lemma 1. *If χ is nonprincipal of level N, then*

$$\chi(1) + \cdots + \chi(N) = 0.$$

Proof. The sum is invariant under multiplication by $\chi(m)$ for any $m \in \mathbb{N}$ coprime to N, but if χ is nonprincipal, then we can choose m with $\chi(m) \neq 1$. □

Sometimes a Dirichlet character of level N can be written as the termwise product of the principal character of level N with a character of some level $N' < N$ (of course N' must divide N). We say the character is *imprimitive* in this case and *primitive* otherwise.

2 L-series

The Dirichlet series associated to a Dirichlet character χ of level N is called a *Dirichlet L-series* (or *Dirichlet L-function*) *of level* N, denoted $L(s, \chi)$. (It may also be denoted $L_\chi(s)$, so that one can refer to L_χ as a function without explicitly naming the variable.) Since χ is completely multiplicative, $L(s, \chi)$ formally factors as

$$\prod_p (1 - \chi(p)p^{-s})^{-1}. \tag{1}$$

In particular, if χ is imprimitive corresponding to the character χ' of level N', then

$$L(s,\chi) = L(s,\chi') \prod_{p|N, p\nmid N'} (1 - \chi'(p)p^{-s}). \tag{2}$$

(Note that (2) reduces to $L(s,\chi) = L(s,\chi')$ if N and N' have the same prime factors, e.g., if N' is prime and $N = (N')^2$.) In particular, the abscissa of absolute convergence of the principal character of level N, and hence of each of the characters of level N, is 1, and the product representation (1) is valid for $\mathrm{Re}(s) > 1$. In particular, $L(s,\chi) \neq 0$ for $\mathrm{Re}(s) > 1$.

Theorem 2. *Let χ be a Dirichlet character of level N. Then $L(s,\chi)$ extends to a meromorphic function on $\mathrm{Re}(s) > 0$ with no poles away from $s = 1$. If χ is principal, then $L(s,\chi)$ has a simple pole at $s = 1$ of residue $\prod_{p|N}(1 - p^{-1})$; otherwise, $L(s,\chi)$ is holomorphic also at $s = 1$.*

Proof. If χ is principal, then by (2),

$$L(s,\chi) = \zeta(s) \prod_{p|N} (1 - p^{-s}),$$

and the claims about $L(s,\chi)$ follow from what we already know about ζ. So assume hereafter that χ is nonprincipal. By partial summation, we can write

$$L(s,\chi) = \sum_{n=1}^{\infty} (\chi(1) + \cdots + \chi(n))(n^{-s} - (n+1)^{-s}). \tag{3}$$

Since $\chi(1) + \cdots + \chi(N) = 0$ by Lemma 1, the quantities $\chi(1) + \cdots + \chi(n)$ are bounded for all n. Meanwhile,

$$\begin{aligned} n^{-s} - (n+1)^{-s} &= n^{-s}(1 - (1 + 1/n)^{-s}) \\ &= sn^{-s-1} + O(n^{-s-2}), \end{aligned}$$

where the implied constant in the big O can be taken uniform over s in a compact set. Consequently, the sum representation for $L(s,\chi)$ given by (3) converges uniformly for $\mathrm{Re}(s) \geq \epsilon$ for any $\epsilon > 0$. This yields the claim. $\square$

3 Nonvanishing of L-functions on $\mathrm{Re}(s) = 1$

Much as we used nonvanishing of ζ on the line $\mathrm{Re}(s) = 1$ to study the prime number theorem, we will use nonvanishing of L-functions on that line to study the prime number theorem in arithmetic progressions. An additional wrinkle, though, is that we have to do some extra work to understand what is going on at $s = 1$ itself; see next section.

Lemma 3. *Let $f(s)$ be a meromorphic function on a neighborhood of* $\mathrm{Re}(s) \geq L$*, with at worst a simple pole at $s = L$ and no other poles. Suppose that* $\log f(s)$ *is represented by a Dirichlet series with abscissa of convergence $\leq L$ and* nonnegative real *coefficients. Then $f(s) \neq 0$ for* $\mathrm{Re}(s) \geq L$.

Proof. See exercises. $\square$

Theorem 4. *Let N be a positive integer. Let $f_N(s)$ be the product of all of the Dirichlet L-series of level N. Then $f_N(s) \neq 0$ for $s \in \mathbb{C}$ with $\mathrm{Re}(s) = 1$.*

Proof. Note that for $\mathrm{Re}(s) > 1$, we have

$$\log f_N(s) = \sum_{p:(p,N)=1} \sum_{n=1}^{\infty} \left(\sum_{\chi} \chi(p^n) \right) p^{-ns}, \tag{4}$$

which is a Dirichlet series with nonnegative real coefficients. (The sum over χ is invariant under multiplication by $\chi(p^n)$ for any single χ, so either the sum is zero or all of the summands are equal to 1.) We may thus apply Lemma 3. $\square$

This tells us a lot about nonvanishing of individual L-functions, but not quite everything.

Theorem 5. *For any Dirichlet character χ, $L(s, \chi) \neq 0$ when $\mathrm{Re}(s) = 1$ and $s \neq 1$.*

Proof. Let N be the level of χ. Then $f_N(s)$ is a product of functions, one of which is $L(s, \chi)$, all of which are holomorphic at s. By Theorem 4, $f_N(s)$ has no zero at s, so none of the factors can either. $\square$

It will take a bit more work to deal with $s = 1$; see next section.

4 Nonvanishing for L-functions at $s = 1$

At $s = 1$ (the so-called *critical point* for Dirichlet L-functions), life is a bit more complicated; to deduce that none of the $L(1, \chi)$ vanish, I would need to know that the function $f_N(s)$ in Theorem 4 has a simple pole, rather than being holomorphic, at $s = 1$.

We say a Dirichlet character is *real* if it takes values in ± 1, and *nonreal* (or *complex*) otherwise.

Theorem 6. *For any nonreal Dirichlet character χ, $L(1, \chi) \neq 0$.*

Proof. Let N be the level of χ. If $L(1, \chi) = 0$, then also $L(1, \overline{\chi}) = 0$, where $\overline{\chi}$ denotes the complex conjugate character. But then $f_N(s)$ is the product of one factor with a simple pole at $s = 1$ (coming from the principal character), two factors with zeroes at $s = 1$ (coming from χ and $\overline{\chi}$, and a bunch of factors which are holomorphic at $s = 1$. This would force $f_N(s)$ to have a zero at $s = 1$, contradicting Theorem 4. $\square$

For the real characters, the above argument fails because $\overline{\chi}$ and χ are the same character, so they don't give two different contributions to $f_N(s)$. Instead, we use a different trick. (There are a number of proofs of this result; see exercises for a second approach.)

Theorem 7. *For any real nonprincipal Dirichlet character χ, $L(1, \chi) \neq 0$.*

Proof. Assume on the contrary that $L(1,\chi) = 0$. Define

$$\psi(s) = \frac{L(s,\chi)L(s,\chi_0)}{L(2s,\chi_0)},$$

where χ_0 is the principal character of level N. Then the numerator of ψ is holomorphic for $\mathrm{Re}(s) > 0$, because $L(s,\chi)$ counterbalances the simple pole of $L(s,\chi_0)$ at $s = 1$. On the other hand, the denominator of ψ is holomorphic and nonzero for $\mathrm{Re}(s) > 1/2$; moreover, it extends meromorphically to a neighborhood of $s = 1/2$ with a simple pole at $s = 1/2$. Therefore ψ is holomorphic for $\mathrm{Re}(s) > 1/2$, and extends holomorphically to a neighborhood of $1/2$ with a simple zero at $s = 1/2$.

However, $\psi(s)$ admits the formal factorization

$$\psi(s) = \prod_{p:\chi(p)=1} \left(\frac{1+p^{-s}}{1-p^{-s}}\right)$$

and so expands as a Dirichlet series with nonnegative real coefficients and constant coefficient 1. The product factorization converges absolutely for $\mathrm{Re}(s) > 1$, so the Dirichlet series does too. But ψ is holomorphic for $\mathrm{Re}(s) > 1/2$, so Landau's theorem implies that the Dirichlet series converges absolutely on $\mathrm{Re}(s) > 1/2$.

This yields $\psi(s) \geq 1$ for $s > 1/2$, whereas $\lim_{s\to(1/2)^+} \psi(s) = 0$, contradiction. $\square$

The proofs above have the merit that one could rewrite them without using complex analysis, in order to obtain a complex analysis-free proof of Dirichlet's theorem. (Dirichlet was working before the properties of complex analytic functions were completely understood, so his proofs tend to only involve real s.) However, Dani and Sawyer pointed out that you can also argue more directly as follows. Suppose *any* of the $L(s,\chi)$ had a zero at $s = 1$; then $f_N(s)$ would be holomorphic on $\mathrm{Re}(s) > 0$. Since $\log f_N(s)$ has nonnegative real coefficients, so does $f_N(s)$ by formal exponentiation. Landau's theorem would then imply that the Dirichlet series for $f_N(s)$ converges absolutely for $\mathrm{Re}(s) > 0$. However, since the Dirichlet series for $\log f_N(s)$ diverges for $s = 1 - 1/\phi(N)$ (exercise), so does the series for $f_N(s)$, contradiction.

5 Historical aside: Dirichlet's class number formula

Dirichlet introduced the Dirichlet L-series before Riemann had introduced complex function theory into the picture, and so did not have access to such simple arguments in order to prove $L(1,\chi) \neq 0$. However, the workaround he found is quite important in its own right; he was able to express the value $L(1,\chi)$ in terms of a important numerical invariant, the *class number* of binary quadratic forms of a given discriminant. That number evidently being positive, he obtained nonvanishing of $L(1,\chi)$, and even determined its sign.

Nowadays, one typically expresses this in the language of algebraic number theory. (If you are not familiar with this language, feel free to ignore the rest of this section.) Let K

be a quadratic number field, and let χ_K be the character such that

$$\chi_K(p) = \begin{cases} 1 & p \text{ is unramified and split in } K \\ -1 & p \text{ is unramified and inert in } K \\ 0 & p \text{ is ramified in } K. \end{cases}$$

One then proves that $L(1, \chi_K)$ is equal to the class number h_K of K times the regulator R_K. (The latter equals 1 if K is imaginary quadratic, and otherwise is equal to a fixed normalization factor times the logarithm of the fundamental unit of K.)

The point here is that $L(1, \chi_0)L(1, \chi_K)$ is (up to multiplication by Euler factors for the ramified primes) equal to the Dedekind zeta function ζ_K of K, defined by

$$\zeta_K(s) = \sum_{\mathfrak{a}} \mathrm{Norm}(\mathfrak{a})^{-s},$$

for $\mathfrak{a}$ running over nonzero ideals of the ring of integers $\mathfrak{o}_K$. For a general number field K, ζ_K has a simple pole at 1, whose residue is the class number of K times the regulator of K times a normalization factor (determined by the number of real and complex places of K); the point is that each factor in this product is visibly nonzero.

One sometimes turns this around and tries to use analytic information about ζ_K to get information about the product $h_K R_K$. It is quite difficult to separate the two factors in this expression; indeed, one can make a good case that they really are simply two separate factors in the computation of the volume of a certain compact topological group, the *Arakelov class group* of R, whose group of components is isomorphic to the usual class group.

Notable exception: there is no regulator for an imaginary quadratic field, so you can get good bounds in this case. For instance, the Brauer-Siegel theorem says that the class number of an imaginary quadratic field of discriminant D is at least $c_\epsilon D^{1/2-\epsilon}$ for any $\epsilon > 0$, though unfortunately the constant c_ϵ cannot be effectively determined from ϵ. The best effective results are due to Goldfeld, who proves an effective lower bound which is polynomial in $\log(D)$; this is a far cry from the truth, but is for instance enough to solve Gauss's class number 1 problem (there are exactly nine imaginary quadratic fields of class number 1).

Exercises

1. Prove Lemma 3. (Hint: recall how you proved the special case $f = \zeta$ earlier.)

2. Let f be a meromorphic function on some neighborhood of $\mathrm{Re}(s) \geq L$, with a pole of order $e > 0$ at $s = L$ and no other poles. Suppose that $\log f(s)$ is represented by a Dirichlet series with nonnegative real coefficients and abscissa of absolute convergence $\leq L$. Prove that every zero of f on the line $\mathrm{Re}(s) = L$ has multiplicity $\leq e/2$. (For $e = 1$, this implies Lemma 3.)

3. Prove directly that the Dirichlet series in (4) does not converge for $s = 1 - 1/\phi(N)$. (Hint: for every Dirichlet character χ of order N, $\chi^{\phi(N)}$ is principal.)

4. Let χ be a real nonprincipal character. Use Dirichlet's hyperbola method (from a prior homework) to show that

$$\sum_{n \leq x} f(n) n^{-1/2} = 2L(1, \chi) x^{1/2} + O(1),$$

for $f(n) = \sum_{d|n} \chi(d)$.

5. Use the previous exercise to show that $L(1, \chi) > 0$, giving a second proof of Theorem 7. (Hint: prove that $f(n) \geq 1$ if n is a perfect square, and $f(n) \geq 0$ otherwise.)